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Fractal Attractors and Singular Invariant Measures in Two-Sector Growth Models with Random Factor Shares*

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Abstract

We analyze a multi-sector growth model subject to random shocks affecting the two sector-specific production functions twofold: the evolution of both productivity and factor shares is the result of such exogenous shocks. We determine the optimal dynamics via Euler-Lagrange equations, and show how these dynamics can be described in terms of an iterated function system with probability. We also provide conditions that imply the singularity of the invariant measure associated with the fractal attractor. Numerical examples show how specific parameter configurations might generate distorted copies of the Barnsley's fern attractor.

Keywords: Two-sector Growth Model, Stochastic Factor Shares, Iterated Function Sistems Fractal Attractors, Singular Measures.
JEL Classification: C61, O41

1 Introduction

Macroeconomic models, and in particular economic growth models, have attracted large interest over the last few decades because of their ability to generate complicated dynamics (Boldrin and Montrucchio, 1986). It is now well known that such models can also give rise to random dynamics eventually converging to invariant measures supported on fractal sets (Montrucchio and Privileggi, 1999). A growing number of studies has recently focused on characterizing the conditions under which this might occur by relying on the iterated function system (IFS) literature (Hutchinson, 1981; Vrscay, 1991; Barnsley, 1993). Most of the existing works dealing with economic growth and IFS analyze the traditional discrete time one-sector growth model with logarithmic utility and Cobb-Douglas production, in either its simplest setup or in slightly

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extended formulations; such a basic model through an appropriate transformation can be converted into a one-dimensional IFS, and it is thus possible to show that its optimal dynamic path may converge to a singular measure supported on a Cantor set, and also that the invariant probability may be either singular or absolute continuous according to specific parameter configurations (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004, 2006, 2009; Marsiglio, 2012; Privileggi and Marsiglio, 2013). Very few are those that instead consider more sophisticated two-sector growth models giving rise to a two-dimensional IFS; the analysis in this framework is clearly more complicated but it is still possible to show that the optimal dynamic path may converge to a singular measure supported on some fractal set, like the Sierpinski gasket, and to eventually characterize singularity versus absolute continuity of the invariant probability (La Torre et al., 2011; La Torre et al., 2015).

We wish to contribute to this literature by extending the analysis of two-sector random growth models and their relation with fractal steady states in order to allow for the random shock to affect not only the productivity level of the sector-specific production functions (La Torre et al., 2011, 2015), but also their factor shares. To the best of our knowledge, the possibility of exogenous shocks on factor shares thus far have been considered only in the one-sector growth model by Mirman and Zilcha (1975), which has recently been extended to the case of learning by Mirman et al. (2016). Nonetheless, it is an interesting generalization of the traditional setup both from the economic and mathematical point of view; indeed, variable factor shares may describe the change in the structure of economic activities which we have observed in industrialized economies over the last decades (Nickell et al., 2008; Marsiglio et al., 2016), and also imply that the optimal economic dynamics may be characterized by an IFS with variable coefficients which makes the analysis of convergence and invariant probability properties not trivial at all. In order to look at this in the simplest possible setup we build on the model discussed in La Torre et al. (2011) in which endogenous growth is ruled out (see La Torre et al., 2015, for a discussion of how results may differ in a framework with endogenous growth), and show that through an appropriate log-transformation the optimal nonlinear dynamic system can be converted into a topologically equivalent linear IFS, although such a transformation requires us to impose a substantial number of restrictions on the model's parameters. We can however show that the system converges to a singular measure supported on some fractal set, which (because of the imposed restrictions) turns out to be a distorted copy of Barnley's fern. We also provide some sufficient conditions under which the associated self-similar measure may be singular.

The paper proceeds as follows. In Section 2 we briefly summarize some basic results from the IFS theory and present some novel sufficient conditions (Theorem 1) for testing the singularity of the invariant distribution in a two-dimensional IFS setup. In Section 3 we analyze a two-sector economic growth model in which random shocks affect both the productivity and the factor shares of the two sector-specific production functions, and we fully characterize the optimal policies through Euler-Lagrange equations. In Section 4 we introduce a log-transformation which allows us to reduce the nonlinear IFS associated with the optimal dynamics to a topologically equivalent linear IFS, which substantially simplifies our analysis but requires to impose some restrictions on the possible parameter values. We also provide, sufficient conditions for the attractor of this linear IFS to be a fractal set (the Barnsley's fern), and we identify sufficient conditions under which the self-similar measure may turn out to be singular. Section 5 presents a specific example of attractor and in particular it shows that the parameter restrictions required by our log-transformation preclude us from generating the original fern, and thus we can obtain only a distorted copy. Section 6 presents concluding remarks and proposes directions for future research.

2 Iterated Function Systems and Fractal Attractors

Hutchinson (1981) and, shortly thereafter, Barnsley (1993) showed how systems of contractive maps with associated probabilities, referred to as IFS by the latter, can be used to construct fractal, self-similar sets and measures. More in general, the action of a generalized fractal transform (GFT) (Kunze et al., 2012) $T : X \to X$ on an element u of the complete metric space (X, d) can be summarized in the following steps. It produces a set of N spatially-contracted copies of u and then it modifies the values of these copies by means of a suitable range-mapping. Finally, it recombines them using an appropriate operator in order to get the element $v \in X$, $v = Tu$. In all these cases, under appropriate conditions, the fractal transform T is a contraction and thus Banach's fixed point theorem guarantees the existence of a unique fixed point $\bar{u} = T\bar{u}$. The inverse problem is a key factor for applications: given a "target" element $v \in X$, we look for a point-to-point contraction mapping T with fixed point $\bar{u}$ such that $d(v, \bar{u})$ is as small as possible. In practical applications, however, it is difficult to construct solutions to this problem and we generally rely on the following simple consequence of Banach's fixed point theorem, known in the fractal coding literature as the *collage theorem*, which states that

$$d(v, \bar{u}) \leq \frac{1}{1-c} d(v, Tv) \tag{1}$$

(c is the contractivity factor of T). Instead of trying to minimize the error $d(v, \bar{u})$, we look for a contraction mapping T that minimizes the *collage error* $d(v, Tv)$.

2.1 Self-Similar Attractors and Invariant Measures

An *N-map iterated function system* (IFS) (Hutchinson, 1981; Barnsley, 1993) is a set of N contraction maps $w_i : X \to X$, *i.e.*, for each $1 \leq i \leq N$, there exists a $c_i \in [0, 1)$ such that $d(w_i(x), w_i(y)) \leq c_i d(x, y)$ for all $x, y \in X$. Associated with an N-map IFS is a set-valued mapping $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$, where $\mathcal{H}(X)$ denotes the set of nonempty compact subsets of X:

$$\mathbf{w}(S) = \bigcup_{i=1}^{N} w_i(S), \quad S \subset X, \tag{2}$$

In Hutchinson (1981) it was proved that

$$h(\mathbf{w}(A), \mathbf{w}(B)) \leq c h(A, B), \quad \forall\ A, B \in \mathcal{H}(X), \tag{3}$$

where h is the Hausdorff distance between compact sets (Kunze et al., 2012) and $c = \max_{1 \leq i \leq N}\{c_i\}$. From Banach's Fixed Point Theorem, there exists a unique fixed point $A \in \mathcal{H}(X)$, known as the *attractor* of the IFS, which satisfies $A = \mathbf{w}(A)$. From Eq. (2), the attractor A is *self-similar* since it may be expressed as a union of contracted copies of itself. For more details and examples one can refer to (Barnsley, 1993; Kunze et al., 2012).

Let $\mathbf{p} = (p_1, \cdots, p_N)$ denote a set of probabilities associated with the IFS maps $\mathbf{w} = (w_1, \cdots, w_N)$, such that $\sum_{i=1}^{N} p_i = 1$. The result is an *N-map IFS with probabilities*, to be denoted as $(\mathbf{w}, \mathbf{p})$. In what follows, let $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X and $\mathcal{M}(X)$ be the set of probability measures on $B(X)$. Associated with an N-map IFS is an operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$, often referred to as the "Markov operator". Its action on $\mathcal{M}(X)$ is defined as follows: For any $\mu \in \mathcal{M}(X)$,

$$(M\mu)(S) = \sum_{i=1}^{N} p_i\, \mu \circ w_i^{-1}(S), \quad S \subseteq X. \tag{4}$$

Here, $w_i^{-1}(S)$ denotes the pre-image of S under w_i and $\circ$ denotes the composition.

We now consider the following Monge-Kantorovich metric on $\mathcal{M}(X)$,

$$d_H(\mu,\nu) = \sup_{f\in \mathrm{Lip}(X)} \left\{ \int_X f d\mu - \int_X f d\nu \right\}, \quad \mu,\nu \in \mathcal{M}(X)$$

where $\mathrm{Lip}(X) = \{f : X \to R, |f(x) - f(y)| \leq d(x,y), x, y \in X\}$. In the IFS literature, this metric is generally referred to as the "Hutchinson" metric. The metric space $(\mathcal{M}(X), d_H)$ is complete (Hutchinson, 1981; Barnsley, 1993) if X is compact (this conclusion is also true if X is complete and the first moment condition is satisfied; Kunze et al., 2012). Moreover, the Markov operator M defined in Eq. (4) is a contraction mapping on $(\mathcal{M}(X), d_H)$ (Hutchinson, 1981), *i.e.*,

$$d_H(M\mu, M\nu) \leq c d_H(\mu,\nu), \quad \mu,\nu \in \mathcal{M}(X),$$

where c was defined earlier, cf. Eq. (3). From Banach's Theorem, there exists a unique measure $\bar{\mu} \in \mathcal{M}(X)$, the so-called *invariant measure* of the N-map IFS, such that $M\bar{\mu} = \bar{\mu}$. This relation may also be viewed as a self-similarity property of μ, *i.e.*, that it may be expressed as a sum of copies of itself. For any $\mu_0 \in \mathcal{M}(X)$, the sequence $\mu_{n+1} = M\mu_n$ converges to the steady state $\bar{\mu}$ when $n \to +\infty$. Furthermore, the trajectory generated by the chaos game $x_{n+1} = w_i(x_n)$ with probability p_i is dense in the IFS attractor A (Hutchinson, 1981; Brandt, 1986). Finally, the support of $\bar{\mu}$ coincides with the IFS attractor A.

2.2 Singularity of The Invariant Measure

In this section we consider the case of affine IFS with probabilities on $\mathbb{R}^2$. In what follows let $w_i(x) = A_i x + b_i$, for $i = 1, 2, \ldots, N$, and p_i be the associated probabilities. The following result states a sufficient condition to prove the singularity of the invariant measure of an affine IFS.

Theorem 1 *Let $(\mathbf{w}, \mathbf{p}) = \{w_1, w_2, \ldots, w_N; p_1, p_2, ..., p_N\}$ be an affine IFS on $\mathbb{R}^d$ having maps $w_i : \mathbb{R}^d \to \mathbb{R}^d$ defined by $w_i(x) = A_i x + b_i$, for $i = 1, 2, \ldots, N$, and let $p = (p_1, p_2, \ldots, p_N)$ be the associated probability weights. If*

$$|\det(A_1)|^{p_1} |\det(A_2)|^{p_2} \cdots |\det(A_N)|^{p_N} < p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N} \tag{5}$$

then the invariant measure μ^* defined by $(\mathbf{w}, \mathbf{p})$ is singular.

Proof. Let $\alpha_i = |\det(A_i)|$ and θ be such that

$$\alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_N^{p_N} < \theta < p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}$$

and let K be the attractor of the IFS w_i. By re-indexing if necessary we assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N$. Furthermore for convenience we assume that N is even (this is only used in (6) below). For $\sigma \in \{1, 2, \ldots, N\}^{\mathbb{N}}$, define

$$\sigma(n, i) = \#\{j \leq n : \sigma_j = i\}.$$

Fix $k \geq 0$ and consider the set

$$S_n = \{\sigma \in \{1, 2, \ldots, N\}^n : \left| \frac{\sigma(n,i) - np_i}{\sqrt{n}\sqrt{p_i(1-p_i)}} \right| \leq k, i = 1, 2, \ldots, N\},$$

so that S_n is the set of *k-typical sequences of length n*. Then from Chebyshev's inequality we have

$$P(S_n) \geq 1 - N/k^2$$

independent of n. Furthermore, by Theorem 1.3.4 in Roman (1992) we have

$$\#S_n \leq (p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N})^{-n} N^{C\sqrt{n}}$$

for some constant $C > 0$. For simplicity of notation, define $w_\sigma = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}$ for any $\sigma \in \{1, 2, \ldots, N\}^n$. By the definition of S_n for any $\sigma \in S_n$ we have that

$$\begin{aligned} \mathcal{L}(w_\sigma(K)) &= \alpha_1^{\sigma(n,1)} \alpha_2^{\sigma(n,2)} \cdots \alpha_N^{\sigma(n,N)} \mathcal{L}(K) \\ &\leq \mathcal{L}(K) \, (\alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_N^{p_n})^n \left(\frac{\alpha_N}{\alpha_1} \frac{\alpha_{N-1}}{\alpha_2} \cdots \frac{\alpha_{N/2+1}}{\alpha_{N/2}} \right)^{\beta\sqrt{n}}, \end{aligned} \tag{6}$$

where $\beta > 0$ is an appropriate constant. Now, let

$$J_n = \bigcup_{\sigma \in S_n} w_\sigma(K) \subset K.$$

Then $\mu(J_n) = P(S_n) \geq 1 - 3/k^2$ for all n. Furthermore,

$$\begin{aligned} \mathcal{L}(J_n) &\leq \sum_{\sigma \in S_n} \mathcal{L}(w_\sigma(K)) \\ &\leq \mathcal{L}(K) \, (\#S_n) \, (\alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_N^{p_n})^n \left(\frac{\alpha_N}{\alpha_1} \frac{\alpha_{N-1}}{\alpha_2} \cdots \frac{\alpha_{N/2+1}}{\alpha_{N/2}} \right)^{\beta\sqrt{n}} \\ &\leq \mathcal{L}(K) \, (p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N})^{-n} N^{C\sqrt{n}} \theta^n \left(\frac{\alpha_N}{\alpha_1} \frac{\alpha_{N-1}}{\alpha_2} \cdots \frac{\alpha_{N/2+1}}{\alpha_{N/2}} \right)^{\beta\sqrt{n}} \\ &= \mathcal{L}(K) \left(\frac{\theta}{p_1^{p_1} \cdots p_N^{p_N}} \right)^n \gamma^{\sqrt{n}} \to 0 \end{aligned}$$

as $n \to \infty$ since $0 < \theta < p_1^{p_1} \cdots p_N^{p_N}$ (here $\gamma > 0$ is some appropriate constant). Thus μ^* is singular with respect to Lebesgue measure. ■

Example 1 *The classical Barnsley's fern (Barnsley, 1993) is produced by the following affine IFS:*

$$\begin{aligned} w_1(\mathbf{x}) &= A_1\mathbf{x} + b_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad p_1 = 0.01 \\ w_2(\mathbf{x}) &= A_2\mathbf{x} + b_2 = \begin{bmatrix} -0.15 & 0.28 \\ -0.04 & 0.24 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}, \qquad p_1 = 0.07 \\ w_3(\mathbf{x}) &= A_3\mathbf{x} + b_3 = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix}, \qquad p_1 = 0.07 \\ w_4(\mathbf{x}) &= A_4\mathbf{x} + b_4 = \begin{bmatrix} 0.85 & 0.04 \\ 0.26 & 0.85 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1.60 \end{bmatrix}, \qquad p_1 = 0.85, \end{aligned}$$

whose attractor is plotted in Figure 1 by tracing 50000 random iterations[1] *according to the 4 probability values considered,* $p_i \in \{0.01, 0.07, 0.07, 0.85\}$. *In this case it is trivial to apply condition (5) of Theorem 1 to prove that the invariant measure is singular because* $\det(A_1) = 0$.

[1]The Maple 2015 code is available from the authors upon request.

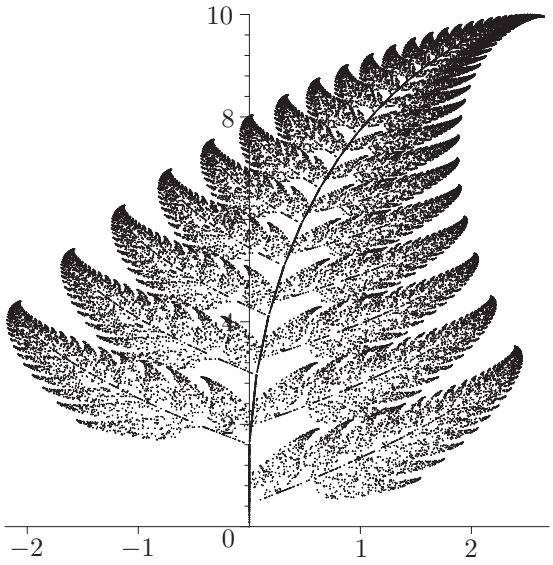

FIGURE 1: approximation thorugh 50000 random iterations of the IFS that generates the classical Barnsley's fern.

3 Economic Growth and Stochastic Factor Shares

We analyze a multi-sector economic growth model subject to random shocks which affect output production. Differently from most studies which assume that shocks affect the total factor productivity (Mitra et al., 2003), we allow for such shocks to affect the factor shares as well, so that shocks have a twofold role in determining economic conditions. In macroeconomic theory factor shares are traditionally assumed to be constant, despite empirical evidence suggests that because of structural changes factor shares tend to be time-varying (Nickell et al., 2008). Very few studies have thus far analyzed the implications of changes in the factor shares on economic dynamics, and they have focused on a framework in which such changes occur deterministically[2] (Growiec et al., 2015; Marsiglio et al., 2016). We contribute to this branch of literature by extending the analysis to a framework in which factor shares evolve randomly, in order to understand what this might imply for macroeconomic dynamics.

For the sake of simplicity, we focus on the standard optimal growth model under uncertainty discussed in La Torre et al. (2011) in which the social planner seeks to maximize the representative household's infinite discounted sum of instantaneous utility functions – which are assumed to be logarithmic – subject to the laws of motion of physical, k_t, and human, h_t, capital. At each time t, the planner chooses consumption, c_t, and the share of human capital, u_t, to allocate into production of the unique homogeneous consumption good which uses a Cobb-Douglas technology combining physical and human capital. Education is assumed to be intensive in human capital, as in Lucas (1988), but the marginal returns of the share of human capital employed in education are decreasing, in accordance with Rebelo (1991). Specifically, the final good and the education sectors are affected by exogenous perturbations which take both a multiplicative form through coefficients z_t and η_t respectively, and an exponential form affecting the factor shares in both production functions; that is, the factor shares in the Cobb-Douglas functions are random as well, so that output at time t is given by $y_t = z_t k_t^{\alpha_t} (u_t h_t)^{\gamma_t}$, where α_t and γ_t denote the physical and human capital factor shares at time t respectively,

[2]In Growiec et al. (2015) factor shares are considered endogenous variables, while in Marsiglio et al. (2016) their change is completely exogenous.

while human capital at time t is produced according to $h_{t+1} = \eta_t \left[(1-u_t)\, h_t\right]^{\phi_t}$, with ϕ_t denoting the time-t human capital factor share in education. The whole $(z_t, \eta_t, \alpha_t, \gamma_t, \phi_t)$ random vector is independent and identically distributed, and can take on m values, *i.e.*, at each time t $(z_t, \eta_t, \alpha_t, \gamma_t, \phi_t) \in \{(z_i, \eta_i, \alpha_i, \gamma_i, \phi_i)\}_{i=1}^{m}$. We shall assume that $(z_i, \eta_i, \alpha_i, \gamma_i, \phi_i) \in \mathbb{R}^5_{++}$ and that $0 < \alpha_i, \gamma_i, \phi_i < 1$, plus $\alpha_i + \gamma_i \leq 1$ for all $i = 1, \ldots, m$. Each vector realization, $(z_i, \eta_i, \alpha_i, \gamma_i, \phi_i)$, occurs with (constant) probability p_i, with $p_i \in (0,1)$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} p_i = 1$.

The social planner problem can thus be summarized as:

$$V(k_0, h_0, z_0, \eta_0, \alpha_0, \gamma_0, \phi_0) = \max_{\{c_t, u_t\}} \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \ln c_t \tag{7}$$

$$\text{s.t.} \begin{cases} k_{t+1} = z_t k_t^{\alpha_t} \left(u_t h_t\right)^{\gamma_t} - c_t \\ h_{t+1} = \eta_t \left[(1-u_t)\, h_t\right]^{\phi_t} \\ k_0 > 0,\ h_0 > 0,\ \ (z_0, \eta_0, \alpha_0, \gamma_0, \phi_0) \text{ are given,} \end{cases} \tag{8}$$

where $\mathbb{E}_0$ denotes expectation at time $t = 0$, $0 < \beta < 1$ is the discount factor. In presence of shocks on the exponents of the Cobb-Douglas production functions it becomes difficult to pursue the usual "guess-and -verify" approach (Bethmann, 2007, 2013; La Torre et al., 2011, 2015) applied to the solution of the Bellman equation, as, already under an i.i.d. assumption on the exogenous shocks, it is not obvious what the best candidate functional form for the value function may look like. Hence, we skip the search of the value function altogether and look directly for the optimal policy by trying to solve the Euler equations. The following Proposition 1 and Theorem 2 provide the basic tools for this procedure.

3.1 Euler-Lagrange Equations and Transversality Condition

Consider the following general reduced-form stochastic intertemporal problem:

$$V(x_0, z_0) = \sup_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \left[u(x_t, x_{t+1}, z_t)\right] \tag{9}$$

$$\text{s.t.} \begin{cases} x_{t+1} \in \Gamma(x_t, z_t) \text{ a.e.} \qquad \forall t \geq 0, \\ x_t \in X \subseteq \mathbb{R}^n,\ z_t \in Z \subseteq \mathbb{R}^s \qquad \forall t \geq 1, \\ x_0 \in X,\ z_0 \in Z \text{ are given,} \end{cases} \tag{10}$$

where $\{z_t\}$ is an i.i.d. process with realizations in $Z \subseteq \mathbb{R}^s$,[3] $\Gamma : X \times Z \to X$ is a correspondence representing the one-period constraint, *i.e.*, it is the set of feasible values for next period's state variable x_{t+1} if the current state is (x_t, z_t). We assume that the set of feasible plans from (x_0, z_0), that is, the set of sequences $x = \{x_t\}$ such that $x_{t+1} \in \Gamma(x_t, z_t)$ a.e. $\forall t \geq 0$, is nonempty; it is well known that a sufficient condition is X to be closed and $\Gamma : X \times Z \to X$ nonempty valued, closed and upper semicontinuous. Moreover, according to Proposition 1, p. 22, and Lemma 1, p. 55, in Hildenbrand (1974), there exist a measurable function $h : X \times Z \to X$ such that $h(x, z) \in \Gamma(x, z)$ for all $(x, z) \in X \times Z$, which allows to define feasible plans recursively: $x_{t+1} = h(x_t, z_t)$ for all $z_t \in Z$ and $t \geq 1$. Clearly, a plan $x = \{x_t\}$ is random, or *contingent*, because it depends on the realization of the stochastic process $\{z_t\}$; in other words, in general different sequences $\{z_t\}$ correspond to different sequences $\{x_t\}$. Indeed, to be precise, we should write $\{x_t(z_{t-1})\}$; we drop the argument z_{t-1} of x_t to simplify notation.[4]

[3] Z can be any Borel measurable subset of $\mathbb{R}^m$, not necessarily a finite set.

[4] Note also that the assumption of $\{z_t\}$ to be an i.i.d. process let x_t to depend a.e. only on the last shock realization, z_{t-1}.

Denote by $G = \{(x, y, z) \in X \times X \times Z : y \in \Gamma(x, z)\}$ the graph of the correspondence $\Gamma(x, z)$, and by $G_z = \{(x, y) \in X \times X : y \in \Gamma(x, z)\}$ the z-section of G. We say that a feasible random plan $x = \{x_t\}$ is *interior* if $(x_t, x_{t+1}) \in \text{Int}(G_{z_t})$ a.e. for all $t \geq 0$. We will also assume the following.

A. 1 *The time $t = 0$ expectation $\mathbb{E}_0[u(x_t, x_{t+1}, z_t)]$ is well defined for all $t \geq 1$, $\text{Int}(X) \neq \varnothing$, and, for each $z \in Z$, the one-period return function $u(\cdot, \cdot, z)$ is differentiable on $\text{Int}(X \times X)$, with each of the first group of n partial derivatives, u_{x_i} for $i = 1, \ldots, n$, absolutely integrable.*

Proposition 1 *Under A.1, if a random plan $x^* = \{x_t^*\}$ is interior and optimal for (9), then it satisfies the following* stochastic Euler-Lagrange equations*:*

$$u_y\left(x_{t-1}^*, x_t^*, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u_x\left(x_t^*, x_{t+1}^*, z_t\right)\right] = 0 \qquad \text{a.e. for all } t \geq 1, \tag{11}$$

where $u_x(\cdot, \cdot, z)$ denotes the vector of partial derivatives of u with respect to the first group of n variables, $u_y(\cdot, \cdot, z)$ denotes the vector of partial derivatives of u with respect to the second group of n variables, and $\mathbb{E}_{t-1}$ denotes expectation at time $t-1$.

Proof. Let $x^* = \{x_t^*\}$ be interior and optimal for (9). Fix a time t, a vector $v \in \mathbb{R}^n$ and consider the new "perturbation" random path

$$\bar{x}^*(\tau) = \left\{x_0, x_1^*, x_2^*, \ldots, x_{t-1}^*, x_t^* + \tau v, x_{t+1}^*, \ldots\right\},$$

where τ is a scalar. Note that $\bar{x}^*(0) = x^*$. Interiority assumption on x^* implies that for $|\tau|$ small enough, say $|\tau| < \varepsilon$, the plan $\bar{x}^*(\tau)$ is feasible, that is, $\left(x_{t-1}^*, x_t^* + \tau v\right) \in G_{z_{t-1}}$ and $\left(x_t^* + \tau v, x_{t+1}^*\right) \in G_{z_t}$ a.e..

Denote the objective as a function of the whole plan $x = \{x_t\}$ by $U(x, z_0) = \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0[u(x_t, x_{t+1}, z_t)]$. As x^* is optimal, $U(x^*, z_0) \geq U[\bar{x}^*(\tau), z_0]$ must hold for $|\tau| < \varepsilon$, which, since the additive perturbation τv affects only the two time-t terms in the sum, is equivalent to

$$\begin{aligned}\beta^{t-1}\mathbb{E}_0\left[u\left(x_{t-1}^*, x_t^*, z_{t-1}\right)\right] + \beta^t \mathbb{E}_0\left[u\left(x_t^*, x_{t+1}^*, z_t\right)\right] \\ \geq \beta^{t-1}\mathbb{E}_0\left[u\left(x_{t-1}^*, x_t^* + \tau v, z_{t-1}\right)\right] + \beta^t \mathbb{E}_0\left[u\left(x_t^* + \tau v, x_{t+1}^*, z_t\right)\right].\end{aligned}$$

Dropping the common term β^{t-1} and noting that, by definition of contingent plan under the assumption that $\{z_t\}$ is an i.i.d. process, x_t^* and $x_t^* + \tau v$ become deterministic whenever the realization z_{t-1} is observed, the last inequality can be rewritten as

$$\begin{aligned}u\left(x_{t-1}^*, x_t^*, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u\left(x_t^*, x_{t+1}^*, z_t\right)\right] \\ \geq u\left(x_{t-1}^*, x_t^* + \tau v, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u\left(x_t^* + \tau v, x_{t+1}^*, z_t\right)\right] \qquad \text{a.e. } \forall\, |\tau| < \varepsilon,\end{aligned}$$

that is, $u\left(x_{t-1}^*, x_t^* + \tau v, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u\left(x_t^* + \tau v, x_{t+1}^*, z_t\right)\right]$ reaches its (interior) maximum in $\tau = 0$ and, under differentiability assumption on $u(\cdot, \cdot, z)$, FOC must hold:

$$\frac{\partial}{\partial \tau}\left\{u\left(x_{t-1}^*, x_t^* + \tau v, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u\left(x_t^* + \tau v, x_{t+1}^*, z_t\right)\right]\right\}\Big|_{\tau=0} = 0 \qquad \text{a.e.},$$

which, since the assumption of absolute integrability of the first n partial derivatives, u_{x_i}, of $u(\cdot, \cdot, z)$ allows the exchange between the differentiation and the expectation operators, boils down to

$$\left\{u_y\left(x_{t-1}^*, x_t^*, z_{t-1}\right) + \beta \mathbb{E}_{t-1}\left[u_x\left(x_t^*, x_{t+1}^*, z_t\right)\right]\right\} \cdot v = 0 \qquad \text{a.e.}.$$

As the last equation holds for all $v \in \mathbb{R}^n$ and $t \geq 1$, we immediately get (11). ■

Problem (9) is said to be *concave* if for each $z \in Z$ then the z-section G_z of the graph G of the correspondence $\Gamma(x, z)$ is convex and the function $u(\cdot, \cdot, z)$ is concave on it.

Theorem 2 *Under A.1, assume that (9) is concave and a plan* $\{x_t^*\}$ *exist such that it is interior and satisfies the stochastic Euler-Lagrange equations (11). If, in addition,* $X \subseteq \mathbb{R}_+^n$, $u_x(x,y,z) \subseteq \mathbb{R}_+^n$, *and the transversality condition*

$$\lim_{t\to\infty} \beta^t \mathbb{E}_0 \left[u_x \left(x_t^*, x_{t+1}^*, z_t \right) \cdot x_t^* \right] = 0 \tag{12}$$

holds, then, $\{x_t^*\}$ *is optimal for (9).*

Proof. Fix $n \geq 1$, let $x^* = \{x_t^*\}$ be an interior plan that satisfies (11) and let $x = \{x_t\}$ be a feasible plan from (x_0, z_0). By concavity and differentiability of $u(\cdot,\cdot,z)$, for any $t \geq 0$

$$u(x_t, x_{t+1}, z_t) \leq u\left(x_t^*, x_{t+1}^*, z_t\right) + u_x\left(x_t^*, x_{t+1}^*, z_t\right) \cdot (x_t - x_t^*) + u_y\left(x_t^*, x_{t+1}^*, z_t\right) \cdot \left(x_{t+1} - x_{t+1}^*\right),$$

that is,

$$u\left(x_t^*, x_{t+1}^*, z_t\right) - u(x_t, x_{t+1}, z_t) \geq u_x\left(x_t^*, x_{t+1}^*, z_t\right) \cdot (x_t^* - x_t) + u_y\left(x_t^*, x_{t+1}^*, z_t\right) \cdot \left(x_{t+1}^* - x_{t+1}\right).$$

Taking expectation (recall that concavity is preserved under integration as established, *e.g.*, in Lemma 9.5, p. 261 in Stokey and Lucas, 1989), discounting and summing up, we see that the difference between the n-step return function evaluated at x^* and x, denoted by $H(n)$, satisfies:

$$\begin{aligned}
H(n) &= \sum_{t=0}^{n-1} \beta^t \mathbb{E}_0 \left[u\left(x_t^*, x_{t+1}^*, z_t\right) \right] - \sum_{t=0}^{n-1} \beta^t \mathbb{E}_0 \left[u(x_t, x_{t+1}, z_t) \right] \\
&= \sum_{t=0}^{n-1} \beta^t \mathbb{E}_0 \left[u\left(x_t^*, x_{t+1}^*, z_t\right) - u(x_t, x_{t+1}, z_t) \right] \\
&\geq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_0 \left[u_x\left(x_t^*, x_{t+1}^*, z_t\right) \cdot (x_t^* - x_t) + u_y\left(x_t^*, x_{t+1}^*, z_t\right) \cdot \left(x_{t+1}^* - x_{t+1}\right) \right] \\
&= u_x(x_0^*, x_1^*, z_0) \cdot (x_0^* - x_0) + \left\{ u_y(x_0^*, x_1^*, z_0) + \beta \mathbb{E}_0 \left[u_x(x_1^*, x_2^*, z_1) \right] \right\} \cdot (x_1^* - x_1) \\
&\quad + \sum_{t=2}^{n-1} \beta^{t-1} \mathbb{E}_0 \left\{ \left[u_y\left(x_{t-1}^*, x_t^*, z_{t-1}\right) + \beta \mathbb{E}_{t-1} \left[u_x\left(x_t^*, x_{t+1}^*, z_t\right) \right] \right] \cdot (x_t^* - x_t) \right\} \\
&\quad + \beta^{n-1} \mathbb{E}_0 \left[u_y\left(x_{n-1}^*, x_n^*, z_{n-1}\right) \cdot (x_n^* - x_n) \right] \\
&= \beta^{n-1} \mathbb{E}_0 \left[u_y\left(x_{n-1}^*, x_n^*, z_{n-1}\right) \cdot (x_n^* - x_n) \right],
\end{aligned}$$

where the third equality is obtained rearranging terms in the third line and recalling that $\mathbb{E}_0(z_t) = \mathbb{E}_0[\mathbb{E}_{t-1}(z_t)]$ for any $t \geq 1$, and the last equality holds because by definition $x_0^* = x_0$ and all the terms in curly brackets vanish according to (11). Applying again (11) to the last term, the inequality above boils down to

$$\begin{aligned}
H(n) &\geq \beta^{n-1} \mathbb{E}_0 \left[u_y\left(x_{n-1}^*, x_n^*, z_{n-1}\right) \cdot (x_n^* - x_n) \right] \\
&= -\beta^{n-1} \mathbb{E}_0 \left[\beta \mathbb{E}_{n-1} u_x\left(x_n^*, x_{n+1}^*, z_n\right) \cdot (x_n^* - x_n) \right] \\
&= -\beta^n \mathbb{E}_0 \left[u_x\left(x_n^*, x_{n+1}^*, z_n\right) \cdot x_n^* \right] + \beta^n \mathbb{E}_0 \left[u_x\left(x_n^*, x_{n+1}^*, z_n\right) \cdot x_n \right],
\end{aligned}$$

where, since $x_n \in \mathbb{R}_+^n$ and $u_x(x,y,z) \subseteq \mathbb{R}_+^n$, $u_x\left(x_n^*, x_{n+1}^*, z_n\right) \cdot x_n \geq 0$ holds a.e. for all $n \geq 1$, so that the last term turns out to be nonnegative: $\beta^n \mathbb{E}_0 \left[u_x\left(x_n^*, x_{n+1}^*, z_n\right) \cdot x_n \right] \geq 0$. Therefore, taking the limit as $n \to \infty$ in both sides and using the transversality condition (12) in the first term of the last line, we have $\lim_{n\to\infty} H(n) \geq 0$, and the proof is complete. ■

For a proof of both Proposition 1 and Theorem 2 under much more general assumptions (like no restrictions on the random process governing uncertainty and non-differentiability of the one-period return function) see Privileggi (1995).

3.2 The Optimal Policy

We now apply Theorem 2 to problem (7) to explicitly compute the optimal policy. First we need to restate (7) in reduced-form, that is, eliminate controls and keeping only the two state variables as follows:

$$V\left(k_0,h_0,z_0,\eta_0,\alpha_0,\gamma_0,\phi_0\right)=\max_{\{k_t,h_t\}}\sum_{t=0}^{\infty}\beta^t\mathbb{E}_0\ln\left\{z_tk_t^{\alpha_t}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}-k_{t+1}\right\} \tag{13}$$

$$\text{s.t.}\begin{cases}0\leq k_{t+1}\leq z_tk_t^{\alpha_t}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}\\ 0\leq h_{t+1}\leq\eta_th_t^{\phi_t}\\ k_0>0,\ h_0>0,\ \ (z_0,\eta_0,\alpha_0,\gamma_0,\phi_0)\ \text{ are given,}\end{cases} \tag{14}$$

where the control u_t has been eliminated thanks to an invertible dynamic constraint for human capital accumulation in (8). Recall that $\{(z_t,\eta_t,\alpha_t,\gamma_t,\phi_t)\}$ is an i.i.d. process with realizations in the finite set $Z=\{(z_i,\eta_i,\alpha_i,\gamma_i,\phi_i)\}_{i=1}^m\subseteq\mathbb{R}^5_{++}$. The state space is $X=\mathbb{R}^2_+$ and the dynamic constraint (14) is represented by the correspondence $\Gamma:X\times Z\to X$ defined as

$$\Gamma\left(k,h,z,\eta,\alpha,\gamma,\phi\right)=\left\{(k',h')\in\mathbb{R}^2_+:\left(h'\leq\eta h^{\phi}\right)\wedge\left(k'\leq zk^{\alpha}\left[h-\left(h'/\eta\right)^{1/\phi}\right]^{\gamma}\right)\right\},$$

which is clearly nonempty valued, closed, upper semicontinuous and, for each $(z,\eta,\alpha,\gamma,\phi)\in\mathbb{R}^2_{++}\times[0,1]^3$, the $(z,\eta,\alpha,\gamma,\phi)$-section of its graph G is convex. Since, for each $(z_t,\eta_t,\alpha_t,\gamma_t,\phi_t)\in Z$ the one-period return function $u\left(k_t,h_t,k_{t+1},h_{t+1},z_t,\eta_t,\alpha_t,\gamma_t,\phi_t\right)=\ln\left\{z_tk_t^{\alpha_t}\left[h_t-\left(h_{t+1}/\eta_t\right)^{1/\phi_t}\right]^{\gamma_t}-k_{t+1}\right\}$ is concave in $(k_t,h_t,k_{t+1},h_{t+1})$ on the (convex) set $G_{(z_t,\eta_t,\alpha_t,\gamma_t,\phi_t)}$, then problem (13) is concave.

The Euler-Lagrange equations (11) are formed by pairs of partial derivatives, with respect to k_t and h_t respectively, set equal to zero. The FOC with respect to k_t leads to:

$$-\frac{1}{z_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1}-\left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]^{\gamma_{t-1}}-k_t}+\beta\mathbb{E}_{t-1}\left\{\frac{z_t\alpha_tk_t^{\alpha_t-1}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}}{z_tk_t^{\alpha_t}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}-k_{t+1}}\right\}=0. \tag{15}$$

In order to explicitly solve (15) we shall assume that the optimal plan for physical capital is a constant share of output in each period, that is, we assume that $k_{t+1}=sz_tk_t^{\alpha_t}\left[h_t-\left(h_{t+1}/\eta_t\right)^{1/\phi_t}\right]^{\gamma_t}$, with $0<s<1$. Under this assumption, since $k_t=sz_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1}-\left(h_t/\eta_{t-1}\right)^{1/\phi_{t-1}}\right]^{\gamma_{t-1}}\Longrightarrow z_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1}-\left(h_t/\eta_{t-1}\right)^{1/\phi_{t-1}}\right]^{\gamma_{t-1}}=k_t/s$, and by recalling that $\{(z_t,\eta_t,\alpha_t,\gamma_t,\phi_t)\}$ is an i.i.d. process, then (15) boils down to

$$\begin{aligned}\frac{1}{\frac{k_t}{s}-k_t}&=\beta\mathbb{E}_{t-1}\left\{\frac{z_t\alpha_tk_t^{\alpha_t-1}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}}{z_tk_t^{\alpha_t}\left[h_t-\left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}-sz_tk_t^{\alpha_t}\left[h_t-\left(h_{t+1}/\eta_t\right)^{1/\phi_t}\right]^{\gamma_t}}\right\}\\&=\frac{\beta\mathbb{E}_{t-1}\left(\alpha_t\right)}{\left(1-s\right)k_t}=\frac{\beta\mathbb{E}\left(\alpha\right)}{\left(1-s\right)k_t},\end{aligned}$$

where in the second equality k_t has been pulled out of the expectation because, under our assumption, $k_t = sz_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1} - (h_t/\eta_{t-1})^{1/\phi_{t-1}}\right]^{\gamma_{t-1}}$ is a deterministic choice taken at time $t-1$, with all the information available at that moment (including the optimal choice for h_t), and in the last equality we used the i.i.d. assumption on the random variable α_t, so that $\mathbb{E}(\alpha) = \sum_{i=1}^{m} p_i\alpha_i$ is a constant. Then, the Euler equation becomes:

$$\frac{s}{(1-s)\,k_t} = \frac{\beta\mathbb{E}(\alpha)}{(1-s)\,k_t},$$

which yields the (constant) term $s = \beta\mathbb{E}(\alpha)$, so that, given the optimal choice for the human capital h_{t+1} (or, equivalently, $u_t h_t$), the (candidate) optimal policy for the physical capital turns out to be:

$$k_{t+1} = \beta\mathbb{E}(\alpha)\, z_t k_t^{\alpha_t}\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{1/\phi_t}\right]^{\gamma_t} = \beta\mathbb{E}(\alpha)\, z_t k_t^{\alpha_t}\left(u_t h_t\right)^{\gamma_t}, \tag{16}$$

where in the last equality we have recovered the original control formulation for human capital employed in final production. Hence, the optimal choice for physical capital depends not only on the realizations of the shocks z_t, α_t and γ_t at time t, but on the "average" shock on the capital factor share α, $\mathbb{E}(\alpha)$, as well.

The FOC with respect to h_t leads to:

$$-\frac{z_{t-1}\gamma_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1} - \left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]^{\gamma_{t-1}-1}\frac{1}{\phi_{t-1}}\left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}-1}\frac{1}{\eta_{t-1}}}{z_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1} - \left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]^{\gamma_{t-1}} - k_t}$$
$$+\beta\mathbb{E}_{t-1}\left\{\frac{z_t\gamma_t k_t^{\alpha_t}\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t-1}}{z_t k_t^{\alpha_t}\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t} - k_{t+1}}\right\} = 0.$$

By using the optimal policy for physical capital (16) for both terms k_t and k_{t+1}, the last equation becomes

$$\frac{z_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1} - \left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]^{\gamma_{t-1}-1}\frac{\gamma_{t-1}}{\phi_{t-1}}\left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\frac{\eta_{t-1}}{h_t}\frac{1}{\eta_{t-1}}}{z_{t-1}k_{t-1}^{\alpha_{t-1}}\left[h_{t-1} - \left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]^{\gamma_{t-1}}[1-\beta\mathbb{E}(\alpha)]}$$
$$= \beta\mathbb{E}_{t-1}\left\{\frac{z_t\gamma_t k_t^{\alpha_t}\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t-1}}{z_t k_t^{\alpha_t}\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]^{\gamma_t}[1-\beta\mathbb{E}(\alpha)]}\right\},$$

which simplifies into

$$\frac{\frac{\gamma_{t-1}}{\phi_{t-1}}\left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}}{\left[h_{t-1} - \left(\frac{h_t}{\eta_{t-1}}\right)^{\frac{1}{\phi_{t-1}}}\right]h_t} = \beta\mathbb{E}_{t-1}\left\{\frac{\gamma_t}{\left[h_t - \left(\frac{h_{t+1}}{\eta_t}\right)^{\frac{1}{\phi_t}}\right]}\right\}$$

From the original dynamic constraint in (8) we can recover the control variable formulation for human capital and substitute $h_{t-1} - (h_t/\eta_{t-1})^{1/\phi_{t-1}}$ with $h_{t-1}u_{t-1}$ and $h_t - (h_{t+1}/\eta_t)^{1/\phi_t}$ with $h_t u_t$, while also noting that $(h_t/\eta_{t-1})^{1/\phi_{t-1}} = (1-u_{t-1})\,h_{t-1}$, thus obtaining:

$$\frac{\gamma_{t-1}\,(1-u_{t-1})\,h_{t-1}}{\phi_{t-1}h_{t-1}u_{t-1}h_t} = \beta\mathbb{E}_{t-1}\left(\frac{\gamma_t}{h_t u_t}\right),$$

which, again after pulling h_t out of the expectation from the RHS as it is a deterministic choice taken at time $t-1$ with all the information available at that moment (while u_t, representing a decision to be taken at time t, is still unknown at time $t-1$), and simplifying terms, becomes

$$\frac{\gamma_{t-1}\,(1-u_{t-1})}{\phi_{t-1}u_{t-1}} = \beta\mathbb{E}_{t-1}\left(\frac{\gamma_t}{u_t}\right). \tag{17}$$

Under the i.i.d. assumption we can safely assume that the expectation on the RHS is constant, say $\mathbb{E}_{t-1}\,(\gamma_t/u_t) = \mathbb{E}\,(\gamma/u) \equiv C$,[5] and then rearrange the last equation as

$$\frac{\gamma_{t-1}}{u_{t-1}} = \gamma_{t-1} + \beta C\phi_{t-1},$$

which, taking expectations on both terms, turns into

$$\mathbb{E}\left(\frac{\gamma_{t-1}}{u_{t-1}}\right) = \mathbb{E}\left(\frac{\gamma}{u}\right) = C = \mathbb{E}\,(\gamma) + \beta C\mathbb{E}\,(\phi)$$

yielding the expected ratio

$$\mathbb{E}\left(\frac{\gamma}{u}\right) = C = \frac{\mathbb{E}\,(\gamma)}{1-\beta\mathbb{E}\,(\phi)}.$$

Using the last expression for $\mathbb{E}_{t-1}\,(\gamma_t/u_t)$ in (17) the optimal fraction of human capital to be employed in the final good production is immediately obtained:

$$u_t = \frac{[1-\beta\mathbb{E}\,(\phi)]\,\gamma_t}{[1-\beta\mathbb{E}\,(\phi)]\,\gamma_t + \beta\mathbb{E}\,(\gamma)\,\phi_t}, \tag{18}$$

which, as expected, besides the "average" shocks on the human capital shares in final good and human capital production, $\mathbb{E}\,(\gamma)$ and $\mathbb{E}\,(\phi)$ respectively, directly depends on time through the shock realizations γ_t and ϕ_t. Note that, as by assumption $0 < \beta, \gamma_t, \phi_t, \mathbb{E}\,(\gamma), \mathbb{E}\,(\phi) < 1$, $0 < u_t < 1$ as well, as it is supposed to. Hence, the (candidate) optimal policy for the human capital is given by:

$$h_{t+1} = \eta_t\,[(1-u_t)\,h_t]^{\phi_t} = \eta_t\left\{\frac{\beta\mathbb{E}\,(\gamma)\,\phi_t}{[1-\beta\mathbb{E}\,(\phi)]\,\gamma_t + \beta\mathbb{E}\,(\gamma)\,\phi_t}h_t\right\}^{\phi_t}, \tag{19}$$

while, after plugging u_t as in (18) into (16), the (candidate) optimal policy for the physical capital is given by:

$$k_{t+1} = \beta\mathbb{E}\,(\alpha)\,z_t k_t^{\alpha_t}\,(u_t h_t)^{\gamma_t} = \beta\mathbb{E}\,(\alpha)\,z_t k_t^{\alpha_t}\left\{\frac{[1-\beta\mathbb{E}\,(\phi)]\,\gamma_t}{[1-\beta\mathbb{E}\,(\phi)]\,\gamma_t + \beta\mathbb{E}\,(\gamma)\,\phi_t}h_t\right\}^{\gamma_t}, \tag{20}$$

[5] Since the realization γ_t is associated with a unique configuration $(z_t, \eta_t, \alpha_t, \gamma_t, \phi_t)$ of shocks, it is reasonable to assume that the optimal choice for u_t must be the same whenever such configuration is realized.

Finally, we must check whether the policies (19) and (20) satisfy the transversality condition (12). Since

$$u_{k_t}\left(k_t^*, h_t^*, k_{t+1}^*, h_{t+1}^*, z_t, \eta_t, \alpha_t, \gamma_t, \phi_t\right) = \frac{\beta\mathbb{E}(\alpha)}{[1-\beta\mathbb{E}(\alpha)]\,k_t^*}$$
$$u_{h_t}\left(k_t^*, h_t^*, k_{t+1}^*, h_{t+1}^*, z_t, \eta_t, \alpha_t, \gamma_t, \phi_t\right) = \frac{\beta\mathbb{E}(\gamma)}{[1-\beta\mathbb{E}(\phi)]\,h_t^*},$$

(12) in this case is satisfied as the scalar product $u_x\left(x_t^*, x_{t+1}^*, z_t\right)\cdot x_t^*$ turns out to be constant:

$$\begin{aligned}\lim_{t\to\infty}\beta^t\mathbb{E}_0\left[u_x\left(x_t^*, x_{t+1}^*, z_t\right)\cdot x_t^*\right] &= \lim_{t\to\infty}\beta^t\mathbb{E}_0\left\{\frac{\beta\mathbb{E}(\alpha)}{[1-\beta\mathbb{E}(\alpha)]\,k_t^*}k_t^* + \frac{\beta\mathbb{E}(\gamma)}{[1-\beta\mathbb{E}(\phi)]\,h_t^*}h_t^*\right\}\\ &= \lim_{t\to\infty}\beta^{t+1}\frac{[1-\beta\mathbb{E}(\phi)]\,\mathbb{E}(\alpha)+[1-\beta\mathbb{E}(\alpha)]\,\mathbb{E}(\gamma)}{[1-\beta\mathbb{E}(\alpha)]\,[1-\beta\mathbb{E}(\phi)]}\\ &= 0.\end{aligned}$$

We can thus conclude that (19) and (20) are definitely the optimal policies for human and physical capital respectively. Note that, not surprisingly, such optimal policies simplify into those found in La Torre et al. (2011) whenever factor shares are completely deterministic.

4 Log-Transformation

In order to present our following results in the most general form, according to (19) and (20) we may consider the following nonlinear IFS:

$$\begin{cases} k_{t+1} = \Delta_t z_t k^{\alpha_t} h_t^{\gamma_t} \\ h_{t+1} = \Theta_t \eta_t k_t^{\delta_t} h_t^{\phi_t}, \end{cases} \tag{21}$$

which reduces to the model discussed in the previous section whenever:

$$\delta_t \equiv 0$$
$$\Delta_t = \beta\mathbb{E}(\alpha)\left\{\frac{[1-\beta\mathbb{E}(\phi)]\,\gamma_t}{[1-\beta\mathbb{E}(\phi)]\,\gamma_t+\beta\mathbb{E}(\gamma)\,\phi_t}\right\}^{\gamma_t} \tag{22}$$
$$\Theta_t = \left\{\frac{\beta\mathbb{E}(\gamma)\,\phi_t}{[1-\beta\mathbb{E}(\phi)]\,\gamma_t+\beta\mathbb{E}(\gamma)\,\phi_t}\right\}^{\phi_t}. \tag{23}$$

Vector $(z_t, \eta_t, \alpha_t, \gamma_t, \delta_t, \phi_t) \in \mathbb{R}^6_{++}$ is a random vector independent and identically distributed, and can take on m values, *i.e.*, at each time t $(z_t, \eta_t, \alpha_t, \gamma_t, \delta_t, \phi_t) \in \{(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i)\}_{i=1}^m$. Shocks z_t, η_t enter multiplicatively the two Cobb-Douglas production functions in (21), while $\alpha_t, \gamma_t, \delta_t, \phi_t$ represent shocks on the factor shares. We shall assume that $0 \le \alpha_i, \gamma_i, \delta_i, \phi_i < 1$, $\alpha_i + \gamma_i \le 1$ and $\delta_i + \phi_i \le 1$ for all $i = 1, \ldots, m$. Each vector realization, $(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i)$, occurs with (constant) probability p_i, with $p_i \in (0,1)$, $i = 1, \ldots, m$, and $\sum_{i=1}^m p_i = 1$.

Our goal is to establish conditions under which the affine IFS

$$\begin{cases} \varphi_{t+1} = \alpha_t\varphi_t + \gamma_t\psi_t + \zeta_t \\ \psi_{t+1} = \delta_t\varphi_t + \phi_t\psi_t + \vartheta_t, \end{cases}, \tag{24}$$

where the coefficients $\alpha_t, \gamma_t, \delta_t, \phi_t$ are the exponents in the Cobb-Douglas production functions in the original nonlinear IFS (21) and the additive random vector $(\zeta_t, \vartheta_t) \in \mathbb{R}^2$ takes on m values

corresponding to realizations of the multiplicative shocks (z_t, η_t), is topological equivalent of system (21); that is, there is a one-to-one continuous transformation from the dynamics of (k_t, h_t) defined by (21) to those of (φ_t, ψ_t) as in (24). It is useful to rewrite (24) in vector terms as

$$\begin{bmatrix} \varphi_{t+1} \\ \psi_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_t & \gamma_t \\ \delta_t & \phi_t \end{bmatrix} \begin{bmatrix} \varphi_t \\ \psi_t \end{bmatrix} + \begin{bmatrix} \zeta_t \\ \vartheta_t \end{bmatrix}, \tag{25}$$

where

$$Q_t = \begin{bmatrix} \alpha_t & \gamma_t \\ \delta_t & \phi_t \end{bmatrix} \tag{26}$$

is a random 2×2 matrix that, together with the vector $(\zeta_t, \vartheta_t) \in \mathbb{R}^2$, take on m values corresponding to the m shocks realizations.

Consider the one-to-one logarithmic transformation $(k_t, h_t) \to (\varphi_t, \psi_t)$ defined by:

$$\begin{cases} \varphi_t = \rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3 \\ \psi_t = \rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6. \end{cases} \tag{27}$$

We aim at establishing conditions on parameters $\{(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i, \zeta_i, \vartheta_i)\}_{i=1}^m$ under which coefficients $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$, exist such that (27) defines a one-to-one transformation from the dynamics of (k_t, h_t) defined by (21) to those of (φ_t, ψ_t) as in (24).

We start without further assumptions on $\{(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i, \zeta_i, \vartheta_i)\}_{i=1}^m$ besides $z_i > 0, \eta_i > 0$, $0 \leq \alpha_i, \gamma_i, \delta_i, \phi_i < 1$, $\alpha_i + \gamma_i \leq 1$ and $\delta_i + \phi_i \leq 1$ for all $i = 1, \ldots, m$; then we will add any further restriction at whatever step is required.

Use (27) to rewrite both sides of (24):

$$\begin{cases} \rho_1 \ln k_{t+1} + \rho_2 \ln h_{t+1} + \rho_3 = \alpha_t (\rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3) + \gamma_t (\rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6) + \zeta_t \\ \rho_4 \ln k_{t+1} + \rho_5 \ln h_{t+1} + \rho_6 = \delta_t (\rho_1 \ln k_t + \rho_2 \ln h_t + \rho_3) + \phi_t (\rho_4 \ln k_t + \rho_5 \ln h_t + \rho_6) + \vartheta_t. \end{cases}$$

Then, use (21) to rewrite the LHS in each equation above in order to obtain the following two equations:

$$\begin{aligned} &\rho_1 \ln \Delta_t + \rho_1 \ln z_t + \rho_1 \alpha_t \ln k_t + \rho_1 \gamma_t \ln h_t + \rho_2 \ln \Theta_t + \rho_2 \ln \eta_t + \rho_2 \delta_t \ln k_t + \rho_2 \phi_t \ln h_t + \rho_3 \\ &\quad = \alpha_t \rho_1 \ln k_t + \alpha_t \rho_2 \ln h_t + \alpha_t \rho_3 + \gamma_t \rho_4 \ln k_t + \gamma_t \rho_5 \ln h_t + \gamma_t \rho_6 + \zeta_t, \end{aligned} \tag{28}$$

$$\begin{aligned} &\rho_4 \ln \Delta_t + \rho_4 \ln z_t + \rho_4 \alpha_t \ln k_t + \rho_4 \gamma_t \ln h_t + \rho_5 \ln \Theta_t + \rho_5 \ln \eta_t + \rho_5 \delta_t \ln k_t + \rho_5 \phi_t \ln h_t + \rho_6 \\ &\quad = \delta_t \rho_1 \ln k_t + \delta_t \rho_2 \ln h_t + \delta_t \rho_3 + \phi_t \rho_4 \ln k_t + \phi_t \rho_5 \ln h_t + \phi_t \rho_6 + \vartheta_t. \end{aligned} \tag{29}$$

As these equations must hold for all $t \geq 0$, under the i.i.d. assumption it is sufficient that they hold for all parameters' values, that is, for all $i = 1, \ldots, m$. Hence, from here on we replace the time index t of each term involving only the model's parameters with the index $i = 1, \ldots, m$, while, clearly, the state variables k_t and h_t remain indexed by t. By equating the corresponding coefficients in the LHS and the RHS, equations (28) and (29) become independent of values taken by the variables $\ln k_t$ and $\ln h_t$; this is equivalent to the following conditions:

$$\begin{cases} \delta_i \rho_2 = \gamma_i \rho_4 \\ \gamma_i \rho_1 = (\alpha_i - \phi_i) \rho_2 + \gamma_i \rho_5 \\ (\alpha_i - \phi_i) \rho_4 + \delta_i \rho_5 = \delta_i \rho_1 \\ \gamma_i \rho_4 = \delta_i \rho_2, \end{cases} \qquad \text{for all } i = 1, \ldots, m,$$

which, as the first and the last equations are the same, is equivalent to the following three equations:

$$\begin{cases} \delta_i \rho_2 = \gamma_i \rho_4 \\ \gamma_i \rho_1 = (\alpha_i - \phi_i) \rho_2 + \gamma_i \rho_5 \\ (\alpha_i - \phi_i) \rho_4 + \delta_i \rho_5 = \delta_i \rho_1 \end{cases} \qquad \text{for all } i = 1, \ldots, m. \tag{30}$$

4.1 Case 1: $\gamma_i = \delta_i = 0$ (Q_t is a Diagonal Random Matrix)

Having in mind the growth model of Section 3.2, this scenario on one hand turns out to be meaningless from the economic point of view because the two policies in (21) are uncoupled, that is, each sector employs exclusively itself as the only input, physical capital to produce physical capital, and human capital to produce human capital. On the other hand it leads to a mathematical contradiction because when $\gamma_i = \delta_i = 0$ for all $i = 1, \ldots, m$, $\mathbb{E}(\gamma) = 0$ as well, which implies that the terms $\ln \Delta_i$ and $\ln \Theta_i$ in (22) and in (23) are not defined:

$$\Delta_i = \beta \mathbb{E}(\alpha) \left\{ \frac{[1 - \beta \mathbb{E}(\phi)] \gamma_i}{[1 - \beta \mathbb{E}(\phi)] \gamma_i + \beta \mathbb{E}(\gamma) \phi_i} \right\}^{\gamma_i} = \beta \mathbb{E}(\alpha) \left\{ \frac{0}{0+0} \right\}^{0}$$
$$\Theta_i = \left\{ \frac{\beta \mathbb{E}(\gamma) \phi_i}{[1 - \beta \mathbb{E}(\phi)] \gamma_i + \beta \mathbb{E}(\gamma) \phi_i} \right\}^{\phi_i} = \left\{ \frac{0}{0+0} \right\}^{\phi_i}.$$

This fact is consistent with findings in La Torre et al. (2011; 2015), where it has been shown the purely symmetric Sierpinski Gasket cannot be obtained as the attractor of a growth model; in fact, such case is represented by an IFS of the type in (25) where Q_i is a purely diagonal constant (*i.e.*, nonrandom) matrix with $\alpha_i = \phi_i \equiv 1/3$.

4.2 Case 2: $\gamma_i > 0$

When $\gamma_i > 0$ for all $i = 1, \ldots, m$, the random matrix Q_t in (26) is either upper diagonal or full.[6] Using the first equation in (30) into the last two equations of (30) and dividing the second one by γ_i yield

$$\begin{cases} \rho_1 = \dfrac{\alpha_i - \phi_i}{\gamma_i} \rho_2 + \rho_5 \\ \dfrac{\alpha_i - \phi_i}{\gamma_i} \delta_i \rho_2 + \delta_i \rho_5 = \delta_i \rho_1. \end{cases}$$

If $\delta_i = 0$ the second equation is always satisfied, while if $\delta_i > 0$ we can divide it by δ_i and get the same equation as the first one. In both cases (30) boils down to only two conditions:

$$\begin{cases} \rho_4 = \dfrac{\delta_i}{\gamma_i} \rho_2 \\ \rho_5 = \rho_1 - \dfrac{\alpha_i - \phi_i}{\gamma_i} \rho_2 \end{cases} \qquad \text{for all } i = 1, \ldots, m. \tag{31}$$

The first one implies that $\rho_4 = 0$ whenever $\delta_i = 0$ for all $i = 1, \ldots, m$, as in the model discussed in Section 3.2 (as well as in models studied by La Torre et al., 2011; 2015). Conditions (31) must hold for each $i = 1, \ldots, m$ in order to let both variables $\ln k_t$ and $\ln h_t$ disappear in equations (28) and (29); therefore, pairing equations (31) with (28) and (29) we obtain a system of $4m$ equations of the form

$$\begin{cases} \rho_4 = \dfrac{\delta_i}{\gamma_i} \rho_2 \\ \rho_5 = \rho_1 - \dfrac{\alpha_i - \phi_i}{\gamma_i} \rho_2 \\ (\ln \Delta_i + \ln z_i) \rho_1 + (\ln \Theta_i + \ln \eta_i) \rho_2 + (1 - \alpha_i) \rho_3 - \gamma_i \rho_6 = \zeta_i \\ (\ln \Delta_i + \ln z_i) \rho_4 + (\ln \Theta_i + \ln \eta_i) \rho_5 + (1 - \phi_i) \rho_6 - \delta_i \rho_3 = \vartheta_i \end{cases} \qquad \text{for all } i = 1, \ldots, m. \tag{32}$$

[6] The symmetric case $\delta_i > 0$ is analogous and can be treated in a similar way as in the following.

(32) is a system of $4m$ linear equations in 6 unknowns: ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, and ρ_6; clearly, any attempt to find a solution when there is more than one state of nature, *i.e.*, when $m \geq 2$, is necessarily doomed to fail, as already with $m = 2$ (32) has 8 equations in 6 unknowns. In other words, once again any hope in solving (32) for any truly random dynamics (at least $m = 2$ distinct states of nature) vanishes from the start; more restrictions on parameters are needed.

4.3 Case 3: Further Restrictions on Parameters' Values

In order to (almost) halve the number of equation in (32) for each i, a natural restriction may originate from the first two equations, *i.e.*, on system (31), by letting them be independent of i, so that they remain only two (fixed) conditions for any number m of shocks realizations. In other words, we now assume that

$$\frac{\delta_i}{\gamma_i} \equiv \bar{\delta} \qquad \text{and} \qquad \frac{\alpha_i - \phi_i}{\gamma_i} \equiv \bar{\alpha} \qquad \text{for all } i = 1, \ldots, m. \tag{33}$$

Under (33) the first two equations in (32), that is, system (31), become independent of the other equations, so that we are left with the following system of $2m + 2$ equations in the 6 unknowns ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, and ρ_6:

$$\begin{cases} \rho_4 = \bar{\delta}\rho_2 \\ \rho_5 = \rho_1 - \bar{\alpha}\rho_2 \\ (\ln \Delta_i + \ln z_i)\,\rho_1 + (\ln \Theta_i + \ln \eta_i)\,\rho_2 + (1 - \alpha_i)\,\rho_3 - \gamma_i\rho_6 = \zeta_i \qquad \text{for all } i = 1, \ldots, m \\ (\ln \Delta_i + \ln z_i)\,\rho_4 + (\ln \Theta_i + \ln \eta_i)\,\rho_5 + (1 - \phi_i)\,\rho_6 - \delta_i\rho_3 = \vartheta_i \qquad \text{for all } i = 1, \ldots, m. \end{cases}$$

After substituting ρ_4 and ρ_5 as in the first two equations into all the others, we obtain a system of $2m$ equations in only 4 unknowns, ρ_1, ρ_2, ρ_3, and ρ_6, of the form

$$\begin{cases} (\ln \Delta_i + \ln z_i)\,\rho_1 + (\ln \Theta_i + \ln \eta_i)\,\rho_2 + (1 - \alpha_i)\,\rho_3 - \gamma_i\rho_6 = \zeta_i \\ (\ln \Delta_i + \ln z_i)\,\bar{\delta}\rho_2 + (\ln \Theta_i + \ln \eta_i)\,(\rho_1 - \bar{\alpha}\rho_2) + (1 - \phi_i)\,\rho_6 - \delta_i\rho_3 = \vartheta_i, \end{cases}$$

which, rearranging terms, is equivalent to

$$\begin{cases} (\ln \Delta_i + \ln z_i)\,\rho_1 + (\ln \Theta_i + \ln \eta_i)\,\rho_2 + (1 - \alpha_i)\,\rho_3 - \gamma_i\rho_6 = \zeta_i \\ (\ln \Theta_i + \ln \eta_i)\,\rho_1 + \left[(\ln \Delta_i + \ln z_i)\,\bar{\delta} - (\ln \Theta_i + \ln \eta_i)\,\bar{\alpha}\right]\rho_2 + (1 - \phi_i)\,\rho_6 - \delta_i\rho_3 = \vartheta_i \end{cases} \tag{34}$$

for all $i = 1, \ldots, m$.

We are thus left with a system of $2m$ linear equations of the type (34) in 4 unknowns: ρ_1, ρ_2, ρ_3 and ρ_6. Clearly, any attempt to find a solution for (34) when there are more than two states of nature, *i.e.*, $m > 2$, is again doomed to fail. However, now there exist a minimal configuration for the states of nature, $m = 2$, to represent a truly random scenario. Actually, $m = 2$ is the only case in which system (34) is truly random, is linear and may have a solution for ρ_1, ρ_2, ρ_3 and ρ_6 independent of the values for the parameters $(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i, \zeta_i, \vartheta_i)$, that is, for any arbitrary choice of the 16 values $\{(z_i, \eta_i, \alpha_i, \gamma_i, \delta_i, \phi_i, \zeta_i, \vartheta_i)\}_{i=1}^{2}$.

As our goal is to pursue the construction of fractals on the plane $\mathbb{R}^2$, such scenario is clearly insufficient, as at least 3 different values (shocks) for the multiplicative/ additive constants $z_i, \eta_i, \zeta_i, \vartheta_i$ are required. Therefore, the only feasible option we are left with seems to be adding more constraints on the parameters' values that are still free, namely $z_i, \eta_i, \zeta_i, \vartheta_i$. The idea is to increase the number of unknowns in terms of values $z_i, \eta_i, \zeta_i, \vartheta_i$ in order to let (34) have always $2m$ equations in $2m$ unknowns. Because we want to have enough degrees of freedom on the choice of the shape of the attractor to which the affine IFS (24) [or (25)] eventually converges,

we actually leave such constraints affect only the original multiplicative shocks' values z_i, η_i, while we keep control, except for $\gamma_i > 0$ and conditions (33), on the parameters $\alpha_i, \gamma_i, \delta_i, \phi_i, \zeta_i, \vartheta_i$ that fully define the IFS (24)/ (25).

Since for $m = 2$ system (34) behaves well and any additional state of nature adds, on one hand, two more equations in (34) and, on the other hand, one more pair of values for the multiplicative shocks z_i, η_i, we may think of any number m of shocks realizations and system (34) having $4 + 2(m-2) = 2m$ unknowns, of which the first 4 are the usual ρ_1, ρ_2, ρ_3 and ρ_6, while the remaining $2m - 4$ are a subset of the total $2m$ pairs (z_i, η_i)-values [or, equivalently, $(\ln z_i, \ln \eta_i)$-values]. According to this method system (34) always has $2m$ equations and $2m$ unknowns, of which $2m - 4$ are pairs of multiplicative shocks values. The price to be paid, however, is that, whenever $m > 2$, system (34) ceases to be linear, as the new unknowns (even in their log-expression) $\ln z_i, \ln \eta_i$ enter multiplicatively the other unknowns of the type ρ_i. Therefore, in order to solve (34) when $m > 2$ we necessarily must exploit numerical methods.

5 An Example of (Distorted) Barnsley's Fern

The original Barnsley's (1993) fern is produced by the IFS

$$\begin{bmatrix} \varphi_{t+1} \\ \psi_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_t & \gamma_t \\ \delta_t & \phi_t \end{bmatrix} \begin{bmatrix} \varphi_t \\ \psi_t \end{bmatrix} + \begin{bmatrix} \zeta_t \\ \vartheta_t \end{bmatrix},$$

where, for $m = 4$, according to example 1 the random matrix coefficients, additive constants, and probability values are reported in Table 1.

i	α_i	γ_i	δ_i	ϕ_i	ζ_i	ϑ_i	p_i
1	0	0	0	0.16	0	0	0.01
2	−0.15	0.28	−0.04	0.24	0	0.44	0.07
3	0.20	−0.26	0.23	0.22	0	1.60	0.07
4	0.85	0.04	0.26	0.85	0	1.60	0.85

TABLE 1: coefficients, additive constants, and probability values for the IFS generating the classical Barnsley's (1993) fern.

Clearly such values are quite distant from all the restrictions we introduced in the previous sections. To begin with, we set $\beta = 0.96$. From Section 3 we learn that we must get rid of the negative values for α_2, γ_3 and δ_2 and set $\delta_i \equiv 0$ for all $i = 1, \ldots, 4$ in the fourth column; next, according to Subsections 4.2 and 4.3 we must choose a nonzero value for γ_1 and find values for α_i, γ_i and ϕ_i that satisfy (33). Assuming $\delta_i \equiv \bar{\delta} \equiv 0$ and $\alpha_i = \phi_i$ for all $i = 1, \ldots, 4$ leaves parameter γ_i free and simplifies things quite a bit; as a result, $\bar{\delta} \equiv \bar{\alpha} \equiv 0$ in (33). The following Table 2 contains our tentative choice for parameters' values that try to resemble the table above as much as possible.

As a matter of fact, replacing parameters α_2, γ_3 with positive values which are the opposite of the original ones makes the most damage to the original fern, as it destroys its symmetry. Under these parameters' choice,

$$\mathbb{E}(\alpha) = \mathbb{E}(\phi) = \sum_{i=1}^{4} \alpha_i p_i = 0.7549 \qquad \text{and} \qquad \mathbb{E}(\gamma) = \sum_{i=1}^{4} \gamma_i p_i = 0.0719,$$

i	α_i	γ_i	δ_i	ϕ_i	ζ_i	ϑ_i	p_i
1	0.16	0.01	0	0.16	0	0	0.01
2	0.24	0.28	0	0.24	0	0.44	0.07
3	0.20	0.26	0	0.20	0	1.60	0.07
4	0.85	0.04	0	0.85	0	1.60	0.85

TABLE 2: our selection for the coefficients, additive constants, and probability values used in the IFS (25).

such that the coefficients (22) and (23) defining the optimal policy of the model described in Subsection 3.2 become:

i	Δ_i	Θ_i
1	0.7131	0.9650
2	0.6863	0.6599
3	0.6922	0.6946
4	0.6731	0.8640

According to the last arguments in Subsection 4.3 we must choose two pairs of (z_i, η_i)-values [or, equivalently, $(\ln z_i, \ln \eta_i)$-values] in order to solve the system of 8 equations defined by (34) in the coefficients ρ_1, ρ_2, ρ_3 and ρ_6 and the remaining two pairs of (z_i, η_i)-values [or, equivalently, $(\ln z_i, \ln \eta_i)$-values]. By (arbitrarily) choosing

$$\ln z_1 = \ln \eta_1 = -1 \qquad \text{and} \qquad \ln z_2 = \ln \eta_2 = -0.5, \tag{35}$$

we find the following unique solution for system (34) by means of the standard (symbolic, not numerical) 'solve' routine in Maple 2015:

$$\begin{aligned} \rho_1 &= \rho_5 = 20.7061 \\ \rho_2 &= -0.1193 \\ \rho_3 &= 33.1413 \\ \rho_4 &= 0 \\ \rho_6 &= 25.5279, \end{aligned}$$

with all multiplicative shocks configurations:

i	z_i	η_i
1	0.3679	0.3679
2	0.6065	0.6065
3	0.5503	0.5801
4	1.2268	1.0393

where the first two lines correspond to the choice in (35) and the other two lines are found as a solution of (34).

The resulting attractor of the IFS (25) obtained under our choice of parameters' values in Table 2 is plotted in Figure 2(a), while Figure 2(b) reports the attractor of the corresponding nonlinear IFS (21). Both are obtained by tracing 50000 random interations[7] of the IFS(25) and (21) respectively according to the 4 probability values considered, $p_i \in \{0.01, 0.07, 0.07, 0.85\}$; note the difference in scale induced on the former by the log-transformation (27) applied to the latter. As anticipated before, the symmetry exhibited by the original Barnsley fern in Figure 1

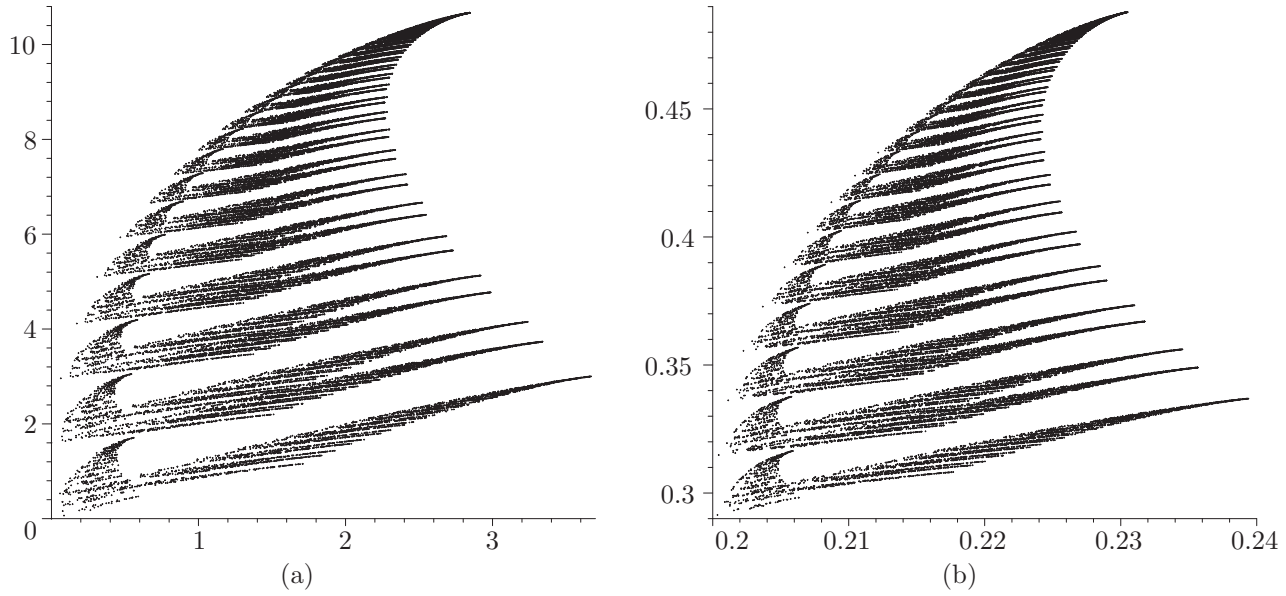

FIGURE 2: approximation thorugh 50000 random iterations of (a) the IFS in (25) and (b) the IFS in (21) for our parameters' values.

is being completely lost because of the choice to replace parameters α_2, γ_3 with opposite values than those in Table 1.

Finally, by applying condition (5) of Theorem 1 to the random matrix Q_t defined in (26) we can easily check that, for the parameters' values reported in Table 2, the invariant measure supported on the attractor in Figure 2(a) turns out to be singular, as

$$|\det(Q_1)|^{p_1} |\det(Q_2)|^{p_2} |\det(Q_3)|^{p_3} |\det(Q_4)|^{p_4} < p_1^{p_1} p_2^{p_2} p_3^{p_3} p_4^{p_4}$$

becomes

$$0.4780 < 0.5732.$$

6 Conclusions

We extend the analysis of stochastic discrete-time optimal growth models to consider a two-sectors framework in which the sector-specific production functions are subject to random shocks affecting not only their productivity but also their factor shares. This extension is interesting both from an economic and mathematical point of view, since it describes the potential structural changes affecting modern economies and it gives rise to an IFS with variable coefficients. We build on the model presented in La Torre et al. (2011) and show that through Euler-Lagrange equations it is possible to characterize the optimal dynamics despite the fact that factor shares are time-varying. Through an appropriate log-transformation we convert the associated nonlinear IFS into a topologically equivalent linear IFS characterized by random coefficients, and this allows us to show that the system converge to a singular measure supported on some fractal set, which (because of the parameter restrictions imposed by our log-transformation) turns out to be a distorted copy of Barnley's fern. We also provide some sufficient conditions under which the associated self-similar measure may be singular.

[7]The Maple 2015 code is avaiable from the authors upon request.

This paper contributes to the stochastic growth literature and fractal attractors by presenting some interesting new results, but it also opens new questions for future research. In particular, extending the analysis in order to characterize also absolute continuity of the invariant measure in a two-dimensional linear IFS, but also singularity versus absolute continuity directly in the original nonlinear two-dimensional IFS might provide some additional insights on the relation between macroeconomic dynamics and fractal attractors. These further issues are left for future research.

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